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Conditional Probabilities and Collapse in Quantum Measurements

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Abstract We show that including both the system and the apparatus in the quantum description of the measurement process, and using the concept of conditional probabilities, it is possible to deduce the statistical operator of the system after a measurement with a given result, which gives the probability distribution for all possible consecutive measurements on the system. This statistical operator, representing the state of the system after the first measurement, is in general not the same that would be obtained using the postulate of collapse.

Keywords Quantum measurements · Projection postulate · Conditional probabilities · Consecutive measurements

1 Introduction

As the measuring instruments are formed by the same kind of matter than everything else, it seems natural to describe the measurement process by quantum theory [1, 2]. This was not the approach of Bohr, who understood the measurement as a primitive notion, having a purely classical description [3]. The first attempt to use quantum theory to investigate the measurement process was due to von Neumann [4]. The quantum interaction establishes a correlation between the macroscopic pointer variables of the apparatus and the microscopic variables of the measured system. In general the final state of the composed system obtained using the Schrödinger equation is a linear superposition of macroscopically distinguishable values of the pointer variable. For those who interpret that this state represents an instrument

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having simultaneously different pointer positions, it is not clear how to relate this final composed state with the definite pointer position that is perceived as a result of an actual single measurement. This difficulty is generally named "the measurement problem". The collapse of the state vector, either postulated or obtained from the addition of non linear terms to the Schrödinger equation, was an attempt to solve this problem. L. Ballentine [7] pointed out the inconsistencies of the collapse postulate with the predictions of ordinary quantum theory, and a recent paper by M. Schlosshawer [8] discuss how ordinary quantum mechanics, with decoherence, can be successfully used to avoid the addition of non linear terms to Schrödinger equation. Moreover, N.G. Van Kampen [5] and latter G. Sewell [6] stressed the importance of the macroscopic character of the measurement instrument to deal with the measurement problem.

We do not try in this paper to give a solution to the "measurement problem" modifying the Schrödinger equation to produce some kind of collapse. On the contrary, we intend to *deduce* the state in which the system is prepared after a measurement with a given result, from the usual quantum formalism applied to the interaction system-apparatus.

A defined choice of the interpretation for the state vector is unavoidable to make contact between the mathematics of quantum theory and the results of the experiments. In this paper the states of the systems are considered as probability distributions, and the state vector is the mathematical tool to compute these probabilities with the Born rule [9–12]. The probabilities, and therefore the state vectors, are properties of an *ensemble* of systems. By the law of large numbers these probabilities are related to the frequencies of results for a big *assembly* of identically prepared experiments [13]. Moreover, in this interpretation, the defined values of individual measurements are assumed as primitive notions.

In Sect. 2 we deduce de collapse of the wave function for the case of ideal measurements. In Sect. 3 we consider non ideal measurements and we show that the collapse postulate is not verified. In Sect. 4 we deduce the defining properties of a generalized measurement from considering the measurement as a quantum process. The macroscopic character of the measurement instrument was considered in Sect. 5. In Appendix we give a short description of the logic of the measurement instruments, which is used through the paper to describe probabilities for consecutive measurements.

2 Ideal Measurements and Collapse

The ideal measurement of an observable Q is an interaction between the system S and the instrument A, which is represented by the following unitary transformation in the Hilbert space $\mathcal{H}_S \otimes \mathcal{H}_A$

$$|\phi\rangle|a_0\rangle \longrightarrow \sum_q \langle q|\phi\rangle|q\rangle|a_q\rangle,$$

where $|q\rangle$ is an eigenvector of the operator $\widehat{Q}$ with eigenvalue q, $|a_0\rangle$ is the initial state of the instrument A and $|a_q\rangle$ is the state of the instrument correlated with the state $|q\rangle$ of the system. The states of the instrument are eigenvectors of a pointer observable $\widehat{A}$ ($\widehat{A}|a_q\rangle = a_q|a_q\rangle$, $\widehat{A} : \mathcal{H}_A \to \mathcal{H}_A$). For simplicity we have not explicitly included in the description the huge number of microscopic variables which together with the pointer define the state of the measurement instrument. This case will be considered in Sect. 5.

The ideal measurement of another observable R requires a different instrument B, and it is represented by a transformation in the corresponding space $\mathcal{H}_S \otimes \mathcal{H}_B$

$$|\phi\rangle|b_0\rangle \longrightarrow \sum_r \langle r|\phi\rangle|r\rangle|b_r\rangle,$$

where $|r\rangle$ is an eigenvector of the operator $\widehat{R}$ with eigenvalue r, $|b_0\rangle$ is the initial state of the instrument B and $|b_r\rangle$ is the state of the instrument correlated with the state $|r\rangle$ of the system. The states of the instrument are eigenvectors of a pointer observable $\widehat{B}$ ($\widehat{B}|b_r\rangle = b_r|b_r\rangle$, $\widehat{B}:\mathcal{H}_B \to \mathcal{H}_B$).

The consecutive measurements of the observables Q and R are represented by consecutive transformations in the composed Hilbert space $\mathcal{H}$ of the system S and instruments A and B ($\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_A \otimes \mathcal{H}_B$),

$$\begin{aligned}
|\Psi_{initial}\rangle &= |\phi\rangle|a_0\rangle|b_0\rangle \\
&\longrightarrow \sum_q \langle q|\phi\rangle\, |q\rangle\, |a_q\rangle|b_0\rangle = \sum_r \sum_q \langle q|\phi\rangle\langle r\,|q\rangle|r\rangle\, |a_q\rangle|b_0\rangle \\
&\longrightarrow \sum_r \sum_q \langle q|\phi\rangle\langle r\,|q\rangle|r\rangle\, |a_q\rangle|b_r\rangle = |\Psi_{final}\rangle.
\end{aligned} \tag{1}$$

The propositions of a classical logic have the structure of an orthocomplemented and distributive lattice [14]. A classical logic can be obtained for the propositions involving the pointer positions of both measurement instruments. For these propositions the usual expressions of the theory of probabilities are valid, particularly those corresponding to conditional probabilities (see Appendix). The use of conditional probabilities to obtain the state of a system prepared by a measurement was given by W.M. de Muynck (see Sect. 3.3.4 of reference [11]).

The probability of measuring the value r of the observable R with the second instrument B, *conditional* on having obtained the value q of the observable Q with the first instrument A, is given by

$$\Pr(b_r|a_q) = \frac{\Pr(b_r \wedge a_q)}{\Pr(a_q)} = \frac{\langle\Psi_{final}|(\widehat{I}_S \otimes |a_q\rangle\langle a_q| \otimes |b_r\rangle\langle b_r|)|\Psi_{final}\rangle}{\langle\Psi_{final}|(\widehat{I}_S \otimes |a_q\rangle\langle a_q| \otimes \widehat{I}_B)|\Psi_{final}\rangle}, \tag{2}$$

where $\widehat{I}_S$ and $\widehat{I}_B$ are the identity operators in the Hilbert spaces $\mathcal{H}_S$ and $\mathcal{H}_B$ and we have used the Born rule for computing the probabilities $\Pr(b_r \wedge a_q)$ and $\Pr(a_q)$. Taking into account the expression for the final state given by (1), it is straightforward to prove that the conditional probability given by (2) can be written in the following simple way

$$\Pr(b_r|a_q) = \langle q|r\rangle\langle r|q\rangle.$$

Moreover, if we consider the projector operator $\widehat{\Pi}_r \equiv |r\rangle\langle r|$ corresponding to the proposition $r = R$, and if we *define* $\widehat{\rho}_q \equiv |q\rangle\langle q|$, the conditional probability can be given the expression

$$\Pr(b_r|a_q) = \mathrm{Tr}[\widehat{\rho}_q \widehat{\Pi}_r]. \tag{3}$$

The first term refer to the probability of certain values of the pointer positions of the instruments A and B, while the second term is written in terms of vectors and operators of the Hilbert space $\mathcal{H}_S$ of the system S.

If we perform a different sequence of measurements on the system, maintaining the first instrument measuring the observable Q, but changing the second instrument for one suitable to the ideal measurement of the observable R', we will obtain

$$\Pr(b'_r|a_q) = \mathrm{Tr}[\widehat{\rho}_q \widehat{\Pi}'_r], \tag{4}$$

where $\widehat{\Pi}'_r \equiv |r'\rangle\langle r'|$ is the projector corresponding to the proposition $R' = r'$.

Equations (3) and (4) give the probabilities to obtain the result r for the measurement of the observable R and the result r' for the observable R', respectively. Therefore, the presence of the corresponding projectors $\widehat{\Pi}_r$ and $\widehat{\Pi}'_r$ in the second terms. Moreover, in both cases, the probabilities are *conditional* to have previously obtained the result q from the measurement of the observable Q. In other words, in both cases the measurements of R and R' are performed on an ensemble of systems S for which the result q of the observable Q was previously obtained.

Equations (3) and (4) also show that this special ensemble of systems is represented by the state operator $\widehat{\rho}_q \equiv |q\rangle\langle q|$. It is evident that this state operator is suitable to compute the probabilities for the values of *any* observable of the system, for the ensemble of systems in which a previous ideal measurement of the observable Q has given the value q.

The initial state of the system is represented by the vector $|\phi\rangle \in \mathcal{H}_S$, while after the measurement it is represented by the vector $|q\rangle \in \mathcal{H}_S$, the eigenvector of the operator $\widehat{Q}$ with eigenvalue q. This result would also have been obtained by using the collapse postulate.

However we did not use the collapse postulate to obtain the result. It was obtained using (i) Schrödinger equation for the unitary evolution given in (1) of the state vector corresponding to the closed system formed by the system S and the instruments A and B, and (ii) conditional probability *defined* by (2) as a quotient of probabilities obtained from the Born rule.

The transformation $|\phi\rangle \to |q\rangle$ of the state of the system S due to the measurement has some remarkable properties which make it very different from the transformations generated by the Schrödinger equation:

(i) it is not a unitary transformation (different states $|\phi\rangle$ and $|\phi'\rangle$ may evolve into the same state $|q\rangle$);
(ii) the transformation $|\phi\rangle \to |q\rangle$ do not represent the evolution of a single ensemble of systems ($|q\rangle$ represents the state of a subensemble of the ensemble unitarily evolved from the state $|\phi\rangle$).

For the case of an ideal measurement, this transformation coincides with the one provided by the collapse postulate, but we have avoided to use this postulate. In our approach the measurement is analyzed as a process fully described by quantum theory. The non-unitary transformation $|\phi\rangle \to |q\rangle$ was deduced from the unitary evolution generated by the Schrödinger equation describing the interaction system-apparatus.

The case of an ideal measurement of an observable with degenerate spectrum can also be obtained in this approach. Let us consider an observable represented by the operator

$$\widehat{Q} = \sum_q q\,\widehat{\Pi}_q, \quad \widehat{\Pi}_q = \sum_{j=1}^{n_q} |q,j\rangle\langle q,j|, \quad \langle q,j|q',j'\rangle = \delta_{qq'}\delta_{jj'},$$

where n_q is the dimension of the subspace of $\mathcal{H}_S$ corresponding to the eigenvectors of $\widehat{Q}$ with eigenvalue q. Any vector $|\phi\rangle \in \mathcal{H}_S$ can be written in terms of the projectors $\widehat{\Pi}_q$

$$|\phi\rangle = \sum_q \sum_{j=1}^{n_q} c_{qj}|q,j\rangle = \sum_q \widehat{\Pi}_q|\phi\rangle,$$

where $c_{qj} \equiv \langle q,j|\phi\rangle$.

An ideal measurement of this observable by an instrument A is represented by the following unitary transformation in $\mathcal{H}_S \otimes \mathcal{H}_A$

$$|q,j\rangle|a_0\rangle \longrightarrow |q,j\rangle|a_q\rangle.$$

After the interaction with instrument A, the system S interacts with another instrument B, making an ideal measurement of an observable represented by the operator $\widehat{R} = \sum_r r|r\rangle\langle r|$, having non degenerate spectrum. The second measurement is represented by the transformation $|r\rangle|b_0\rangle \rightarrow |r\rangle|b_r\rangle$.

The consecutive measurements are represented by an unitary transformation in $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_A \otimes \mathcal{H}_B$

$$|\Psi_{initial}\rangle = |\phi\rangle|a_0\rangle|b_0\rangle \longrightarrow \sum_q \widehat{\Pi}_q|\phi\rangle|a_q\rangle|b_0\rangle \longrightarrow \sum_q \sum_r |r\rangle\langle r|\widehat{\Pi}_q|\phi\rangle|a_q\rangle|b_r\rangle.$$

For the probability to obtain b_r in the second measurement if the result of the first one was a_q we obtain in this case

$$\Pr(b_r|a_q) = \frac{\Pr(b_r \wedge a_q)}{\Pr(a_q)} = \mathrm{Tr}[\widehat{\rho}_q \widehat{\Pi}_r],$$

where $\widehat{\Pi}_r = |r\rangle\langle r|$ and

$$\widehat{\rho}_q = \frac{\widehat{\Pi}_q|\phi\rangle\langle\phi|\widehat{\Pi}_q}{\langle\phi|\widehat{\Pi}_q|\phi\rangle},$$

which is the Lüders projection.

3 Non Ideal Measurements

In this case the system is modified by the measurement process, even when the initial state of the system is an eigenstate of the observable to be measured.

The measurement processes on the eigenvectors $|q\rangle$ and $|r\rangle$ of the operators $\widehat{Q}$ and $\widehat{R}$ are described by the following unitary transformations

$$|q\rangle|a_0\rangle \rightarrow |\mu_q\rangle|a_q\rangle, \quad |r\rangle|b_0\rangle \rightarrow |\nu_r\rangle|b_r\rangle,$$

where $|\mu_q\rangle$ and $|\nu_r\rangle$ are different from the initial states $|q\rangle$ and $|r\rangle$.

Consecutive measurements are represented by the transformation

$$\begin{aligned} |\Psi_{initial}\rangle &= |\phi\rangle|a_0\rangle|b_0\rangle \\ &\longrightarrow \sum_q \langle q|\phi\rangle\,|\mu_q\rangle\,|a_q\rangle|b_0\rangle = \sum_r \sum_q \langle q|\phi\rangle\langle r\,|\mu_q\rangle|r\rangle\,|a_q\rangle|b_0\rangle \\ &\longrightarrow \sum_r \sum_q \langle q|\phi\rangle\langle r\,|\mu_q\rangle|\nu_r\rangle\,|a_q\rangle|b_r\rangle = |\Psi_{final}\rangle, \end{aligned}$$

and the probability that the second instrument measures the value r of R *if* the first instrument has measured the value q of Q is given by the conditional probability

$$\begin{aligned} \Pr(b_r|a_q) &= \frac{\Pr(b_r \wedge a_q)}{\Pr(a_q)} = \frac{\langle\Psi_{final}|(\widehat{I_S} \otimes |a_q\rangle\langle a_q| \otimes |b_r\rangle\langle b_r|)|\Psi_{final}\rangle}{\langle\Psi_{final}|(\widehat{I_S} \otimes |a_q\rangle\langle a_q| \otimes \widehat{I_B})|\Psi_{final}\rangle} \\ &= \frac{|\langle q|\phi\rangle|^2|\langle r\,|\mu_q\rangle|^2}{\sum_r |\langle q|\phi\rangle|^2|\langle r\,|\mu_q\rangle|^2} = |\langle r\,|\mu_q\rangle|^2 = \mathrm{Tr}[\widehat{\rho}_q'\widehat{\Pi}_r], \end{aligned}$$

where $\widehat{\Pi}_r = |r\rangle\langle r|$ corresponds to the proposition $r = R$, and we *define* $\widehat{\rho}_q' \equiv |\mu_q\rangle\langle\mu_q|$.

In this case we have shown that the first measurement with result q has prepared the system in the state $\widehat{\rho}_q'$. The effect of the first measurement on the system is in this case the transformation $|\phi\rangle \to |\mu_q\rangle$, which do not coincide with the collapse postulate. This result was previously obtained by L.E. Ballentine [7], who analyzed the limitations of the collapse postulate.

4 Generalized Measurements

Now we consider the most general measurement process [15]. It is described through a collection of *measurement operators* $\{\widehat{M}_m\}$, acting on the Hilbert space $\mathcal{H}_S$ of the system, and satisfying $\sum_m \widehat{M}_m^\dagger \widehat{M}_m = \widehat{I}_S$. The probability to obtain the result m in the measurement on a state $|\phi\rangle$ is $\Pr(m) = \langle\phi|\widehat{M}_m^\dagger \widehat{M}_m|\phi\rangle$, and if the result is m the transformation on the system is

$$|\phi\rangle \longrightarrow \left(\sqrt{\langle\phi|\widehat{M}_m^\dagger \widehat{M}_m|\phi\rangle}\right)^{-1} \widehat{M}_m|\phi\rangle. \tag{5}$$

In this section we are going to prove that these *defining properties* of a generalized measurement can be *deduced* considering the interaction between the system S and a measurement instrument A, represented by a unitary transformation $\widehat{U}$ in the Hilbert space $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_A$. If we denote by $|m\rangle$ the state of the instrument corresponding to the result m, the measurement operators can be deduced from the following expression

$$\begin{aligned}|\Psi_{initial}\rangle = |\phi\rangle|0\rangle &\longrightarrow |\Psi_{final}\rangle = \widehat{U}(|\phi\rangle|0\rangle)\\ &= \sum_m |m\rangle\langle m|\widehat{U}(|\phi\rangle|0\rangle) \equiv \sum_m (\widehat{M}_m|\phi\rangle)|m\rangle.\end{aligned}$$

The probability to obtain the result m can be deduced from the Born rule

$$\Pr(m) = \langle\Psi_{final}|(\widehat{I}_S \otimes |m\rangle\langle m|)|\Psi_{final}\rangle = \langle\phi|\widehat{M}_m^\dagger \widehat{M}_m|\phi\rangle.$$

If two instruments A and B, with measurement operators $\{\widehat{M}_{m_A}\}$ and $\{\widehat{N}_{m_B}\}$, are used for consecutive measurements on a system S, the process is represented by the following two consecutive unitary transformations

$$\begin{aligned}|\Psi_{initial}\rangle = |\phi\rangle|0_A\rangle|0_B\rangle &\longrightarrow \sum_{m_A}(\widehat{M}_{m_A}|\phi\rangle)|m_A\rangle|0_B\rangle\\ &\longrightarrow \sum_{m_A}\sum_{m_B}(\widehat{N}_{m_B}\widehat{M}_{m_A}|\phi\rangle)|m_A\rangle|m_B\rangle = |\Psi_{final}\rangle.\end{aligned}$$

The probability for the instrument B to give the result m_B conditioned for the fact that the instrument A has already given the result m_A is now obtained from the expression of conditional probability and the Born rule

$$\begin{aligned}\Pr(m_B|m_A) &= \frac{\Pr(m_B \wedge m_A)}{\Pr(m_A)}\\ &= \frac{\langle\Psi_{final}|(\widehat{I}_S \otimes |m_A\rangle\langle m_A| \otimes |m_B\rangle\langle m_B|)|\Psi_{final}\rangle}{\langle\Psi_{final}|(\widehat{I}_S \otimes |m_A\rangle\langle m_A| \otimes \widehat{I}_B)|\Psi_{final}\rangle}\end{aligned}$$

$$= \langle \phi_{m_A} | \widehat{N}^{\dagger}_{m_B} \widehat{N}_{m_B} | \phi_{m_A} \rangle,$$

$$|\phi_{m_A}\rangle \equiv \left(\sqrt{\langle \phi | \widehat{M}^{\dagger}_{m_A} \widehat{M}_{m_A} | \phi \rangle} \right)^{-1} \widehat{M}_{m_A} |\phi\rangle.$$

The state $|\phi_{m_A}\rangle$ can be interpreted as the result of a preparation on the system produced when the instrument A registers the value m_A. The postulated generalized collapse defined in (5) is now deduced from Schrödinger equation and Born rule, by considering the measurement instruments as quantum systems. In this section we have shown, once again, that all the properties defining a general measurement can be deduced considering the measurement as a quantum process of interaction between system and instruments, and that there is no need of collapse postulate.

5 Macroscopic Instruments

In the previous sections we have not included the huge number of microscopic variables of the macroscopic measurement instrument. Including these variables, an operator $\widehat{A}$ representing the pointer of an instrument M_1 has a complete set of eigenvectors in the Hilbert space $\mathcal{H}_{M_1}$, satisfying $\widehat{A}|a,m\rangle = a|a,m\rangle$, where a is the pointer variable, and m labels the many other quantum numbers necessary to specify an eigenvector. For the system S, we consider the measurement of an observable represented by an operator $\widehat{Q}$ in the Hilbert space $\mathcal{H}_S$, having a complete set of eigenvectors verifying $\widehat{Q}|q\rangle = q|q\rangle$.

The non ideal measurement process is represented by an unitary transformation in the Hilbert space $\mathcal{H}_S \otimes \mathcal{H}_{M_1}$, defined by

$$|q\rangle|a_0, m\rangle \longrightarrow |a_q; (q, m)\rangle \equiv \sum_{q'm'} u_{qm}^{q'm'} |q'\rangle|a_q, m'\rangle.$$

Following L.E. Ballentine [7], the labels (r, m) in the final vector do not denote eigenvalues, but they keep the memory of the initial state previous to the measurement. The system-instrument state after the measurement is $|a_q; (q, m)\rangle$, having a well defined value a_q of the pointer variable, but in general not well defined values of the remaining variables. The *initial* value q of the system observable Q is correlated with the *final* value a_q of the pointer observable A.

Analogously, the measurement of another observable represented by the operator $\widehat{R}$ in the Hilbert space $\mathcal{H}_S$ of the system S, is made with an instrument M_2 with pointer operator $\widehat{B}$ in the Hilbert space $\mathcal{H}_{M_2}$. The measurement process is represented by an unitary transformation in the Hilbert space $\mathcal{H}_S \otimes \mathcal{H}_{M_2}$

$$|r\rangle|b_0, n\rangle \longrightarrow |b_r; (r, n)\rangle \equiv \sum_{r'n'} v_{rn}^{r'n'} |r'\rangle|b_r, n'\rangle,$$

where $|r\rangle$ is an eigenvector of $\widehat{R}$ in $\mathcal{H}_S$ ($\widehat{R}|r\rangle = r|r\rangle$) and $|b, n\rangle$ is an eigenvector of the pointer observable $\widehat{B}$ in the Hilbert space $\mathcal{H}_B$ ($\widehat{B}|b, n\rangle = b|b, n\rangle$). The index n represents the quantum numbers different from the label b associated to the pointer.

For an initial state $|\phi\rangle = \sum_q c_q |q\rangle$ of the system S ($c_q \equiv \langle q|\phi\rangle$), the consecutive measurement of observables $\widehat{Q}$ and $\widehat{R}$ is represented by the following consecutive transformation in the Hilbert space $\mathcal{H}_S \otimes \mathcal{H}_{M_1} \otimes \mathcal{H}_{M_2}$

$$|\Psi_{initial}\rangle = |\phi\rangle|a_0; m\rangle|b_0; n\rangle \longrightarrow \sum_q c_q |a_q; (q,m)\rangle|b_0, n\rangle$$

$$\longrightarrow |\Psi_{final}\rangle \equiv \sum_q c_q \sum_r (r|a_q; (q,m))|b_r; (r,n)\rangle,$$

where $(r|a_q; (q,m)) \equiv \sum_{q'm'} u_{qm}^{q'm'} \langle r|q'\rangle|a_q, m'\rangle \in \mathcal{H}_{M_1}$.

By straightforward calculations we obtain

$$\begin{aligned}\Pr(b_r|a_q) &= \frac{\Pr(b_r \wedge a_q)}{\Pr(a_q)}\\ &= \frac{\langle\Psi_{final}|[\widehat{I}_S \otimes \sum_{m'} |a_q, m'\rangle\langle a_q, m'| \otimes \sum_{n'} |b_r, n'\rangle\langle b_r, n'|]|\Psi_{final}\rangle}{\langle\Psi_{final}|[\widehat{I}_S \otimes \sum_{m'} |a_q, m'\rangle\langle a_q, m'| \otimes \widehat{I}_{M_2}]|\Psi_{final}\rangle}\\ &= \mathrm{Tr}[\widehat{\rho}(q,m)\widehat{\Pi}_r],\end{aligned}$$

where $\widehat{\Pi}_r \equiv |r\rangle\langle r|$ and

$$\widehat{\rho}(q,m) \equiv \sum_{q'q''} \left(\sum_{\widetilde{m}} u_{qm}^{q'\widetilde{m}} \overline{u}_{qm}^{q''\widetilde{m}} \right) |q'\rangle\langle q''|.$$

The density operator $\widehat{\rho}(q,m)$ represents the state of the system S after the measurement with the instrument M_1 has given the result a_q. We notice in this case an important difference with the results obtained in the previous sections: even for a system S in an initially pure state, the effect of the instrument microscopic variables is to prepare the system in a non pure state.

6 Conclusions

The collapse of the wave function is usually invoqued to justify the existence of a well defined result of a single measurement process.

Our strategy in this paper has been the opposite. First, we gave a full quantum description of the system- instrument interaction for the measurement process. Second, we accepted the experimental evidence that in each individual experiment, the measurement instrument produce a well defined results. Third, we obtained the probabilities for these results using the Born rule.

For two consecutive measurements on the system, the probability distribution of the possible results of the second measurement conditioned to a determined result of the first one, can be computed with the usual expression for conditional probabilities. From this calculations we have been able to deduce which is the state vector representing the system after a measurement with a given result.

The system is *prepared* in a well defined state by the measurement. This state is strongly dependent on the form of the interaction system-apparatus. The obtained result coincides with that of the collapse postulate only for the ideal measurement, and explicit expressions of the prepared state for non ideal and generalized measurements have also been obtained.

In this way we have been able to provide a satisfactory description of the measurement process as a quantum process, in which it is not necessary to postulate additional physical mechanisms like the collapse of the wave function.

Appendix: The Logic of the Measurement Instruments

Several times in this paper we have considered the measurement of an observables Q with an instrument A on a system S, followed by the measurement of another observable R using a second instrument B. The whole process was described by the evolution of a state vector in the Hilbert space $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_A \otimes \mathcal{H}_B$. We labelled by $\widehat{A}$ and $\widehat{B}$ the corresponding pointer operators having eigenvalues a_q and b_p, and eigenvectors $|a_q\rangle$ and $|b_p\rangle$.

The quantum description of the measurement process should prescribe definite values for the probabilities of propositions like "the result on the first instrument was a_q *and* the result on the second instrument is b_p", or "the result on the second instrument is b_p *if* the result on the first instrument was a_q". These propositions involve eigenvalues of the pointer operators $\widehat{A}$ and $\widehat{B}$, acting on Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$. These operators can be lifted to operators acting on the tensor product space $\mathcal{H}$,

$$\widehat{A}_{\mathcal{H}} \equiv \widehat{I}_S \otimes \widehat{A} \otimes \widehat{I}_B, \quad \widehat{B}_{\mathcal{H}} \equiv \widehat{I}_S \otimes \widehat{I}_A \otimes \widehat{B},$$

where $\widehat{I}_S$, $\widehat{I}_A$ and $\widehat{I}_B$ are the identity operators in the spaces H_S, H_A and H_B. It is evident that the lifted operators $\widehat{A}_{\mathcal{H}}$ and $\widehat{B}_{\mathcal{H}}$ commute, and therefore the possible results of the consecutive measurements have the quantum logic of the simultaneous eigenvectors of a set of commuting operators. The relevant aspect of this logic are reviewed in what follows.

Let us consider a complete set of commuting observables, represented by n operators $\widehat{\overline{R}} \equiv (\widehat{R}_1, \dots, \widehat{R}_n)$, having the complete orthonormal eigenvectors $|\overline{r}\rangle = |r_1, \dots, r_n\rangle$ ($\widehat{\overline{R}}\,|\overline{r}\rangle = \overline{r}\,|\overline{r}\rangle$, $\overline{r} \in \mathbb{R}^n$). The proposition "$\overline{r}$ belongs to the set $\Delta^n \subset \mathbb{R}^n$" is represented by the subspace of the Hilbert space generated by the projector $\widehat{\Pi}_{\Delta^n} = \sum_{\overline{r}\in\Delta^n} |\overline{r}\rangle\langle\overline{r}|$. The conjunction and disjunction of two proposition are represented by the intersection and the direct sum of the subspaces. The order relation is the implication, represented by set inclusion. For two propositions $p_1 = \{\overline{r} \in \Delta_1^n\}$ and $p_2 = \{\overline{r} \in \Delta_2^n\}$ we have the following corresponding projectors [16],

$$\begin{aligned}
p_1 &\longrightarrow \widehat{\Pi}_1 = \textstyle\sum_{\overline{r}\in\Delta_1^n} |\overline{r}\rangle\langle\overline{r}|,\\
p_2 &\longrightarrow \widehat{\Pi}_2 = \textstyle\sum_{\overline{r}\in\Delta_2^n} |\overline{r}\rangle\langle\overline{r}|,\\
p_1 \wedge p_2 &\longrightarrow \lim_{k\to\infty}(\widehat{\Pi}_1\widehat{\Pi}_2)^k,\\
p_1 \vee p_2 &\longrightarrow \widehat{I} - \lim_{k\to\infty}[(\widehat{I} - \widehat{\Pi}_1)(\widehat{I} - \widehat{\Pi}_2)]^k,\\
p_1' &\longrightarrow \widehat{I} - \widehat{\Pi}_1.
\end{aligned} \tag{6}$$

The projectors associated with propositions within the basis $\{|\overline{r}\rangle\}$ are commutative

$$\widehat{\Pi}_1\widehat{\Pi}_2 = \sum_{\overline{r}\in\Delta_1^n} |\overline{r}\rangle\langle\overline{r}| \sum_{\overline{r}'\in\Delta_2} |\overline{r}'\rangle\langle\overline{r}'| = \sum_{\overline{r}\in\Delta_1^n \frown \Delta_2^n} |\overline{r}\rangle\langle\overline{r}| = \widehat{\Pi}_2\widehat{\Pi}_1.$$

From these commutation properties simplified expressions are easily obtained for the projectors associated with conjunction and disjunction

$$\begin{aligned}
p_1 \wedge p_2 &\longrightarrow \widehat{\Pi}_1\widehat{\Pi}_2,\\
p_1 \vee p_2 &\longrightarrow \widehat{\Pi}_1 + \widehat{\Pi}_2 - \widehat{\Pi}_1\widehat{\Pi}_2.
\end{aligned}$$

Propositions of the form $p_1 = \{\overline{r} \in \Delta_1^n\}$, $p_2 = \{\overline{r} \in \Delta_2^n\}$ and $p_3 = \{\overline{r} \in \Delta_3^n\}$ are distributive, i.e.

$$p_1 \wedge (p_2 \vee p_3) = (p_1 \wedge p_2) \vee (p_1 \wedge p_3),$$

$$p_1 \vee (p_2 \wedge p_3) = (p_1 \vee p_2) \wedge (p_1 \vee p_3),$$

as can be easily proved by writing the corresponding projectors. Therefore, within a fixed basis, the lattice of propositions is a classical logic. Moreover, within a fixed basis the usual logic of our language is suitable to talk about quantum propositions.

A probability distribution on a lattice is a function from the propositions to the real numbers satisfying

(i) $\Pr(p) \geq 0$, for all propositions p
(ii) $\Pr(p \vee q) = \Pr(p) + \Pr(q)$ for all propositions p and q such that $p \wedge q = \phi$
(iii) $\Pr(I) = 1$ for the unit proposition I.

Probabilities in quantum theory are calculated using the Born rule. For a pure state represented by the vector ψ of the Hilbert space, the probability of a proposition p is given by $\Pr(p) = \langle\psi|\widehat{\Pi}_p|\psi\rangle$, where $\widehat{\Pi}_p$ is the projector associated with the proposition p. We can prove that conditions (i) (ii) and (iii) are satisfied.

To prove condition (i) consider a proposition $p_{\Delta^n} = \{\overline{r} \in \Delta^n\}$, with the corresponding projector $\widehat{\Pi}_{\Delta^n} = \sum_{\overline{r}\in\Delta^n} |\overline{r}\rangle\langle\overline{r}|$ and compute $\Pr(p_{\Delta^n}) = \langle\psi|\widehat{\Pi}_{\Delta^n}|\psi\rangle = \sum_{\overline{r}\in\Delta^n}\langle\psi|\overline{r}\rangle\langle\overline{r}|\psi\rangle = \sum_{\overline{r}\in\Delta^n} |\langle\overline{r}|\psi\rangle|^2 \geq 0$.

To prove (ii) let us consider two disjoint subsets Δ_1^n and Δ_2^n of $\mathbb{R}^n$. Therefore $\widehat{\Pi}_{\Delta_1^n}\widehat{\Pi}_{\Delta_2^n} = 0$, and therefore $p_{\Delta_1^n} \wedge p_{\Delta_2^n} = \phi$. The projector corresponding to the proposition $p_{\Delta_1^n} \vee p_{\Delta_2^n}$ is $\widehat{\Pi}_{\Delta_1^n} + \widehat{\Pi}_{\Delta_2^n} - \widehat{\Pi}_{\Delta_1^n}\widehat{\Pi}_{\Delta_2^n} = \widehat{\Pi}_{\Delta_1^n} + \widehat{\Pi}_{\Delta_2^n}$. The probability of the disjunction is

$$\begin{aligned}\Pr(p_{\Delta_1^n} \vee p_{\Delta_2^n}) &= \langle\psi|(\widehat{\Pi}_{\Delta_1^n} + \widehat{\Pi}_{\Delta_2^n})|\psi\rangle \\ &= \langle\psi|\widehat{\Pi}_{\Delta_1^n}|\psi\rangle + \langle\psi|\widehat{\Pi}_{\Delta_2^n}|\psi\rangle = \Pr(p_{\Delta_1^n}) + \Pr(p_{\Delta_2^n}),\end{aligned}$$

and condition (ii) is verified.

Property (iii) is easily obtained

$$\Pr(p_{\mathbb{R}^n}) = \langle\psi|\widehat{\Pi}_{\mathbb{R}^n}|\psi\rangle = \sum_{\overline{r}\in\mathbb{R}^n}\langle\psi|\overline{r}\rangle\langle\overline{r}|\psi\rangle = \langle\psi|\widehat{I}|\psi\rangle = 1.$$

The probability for the proposition "the observable R_j has the value r_j in the set Δ_j *if* the observable R_i has the value r_i in the set Δ_i" can be *defined* by the standard expression for the conditional probability

$$\begin{aligned}\Pr(p_{\Delta_j}|p_{\Delta_i}) &\equiv \frac{\Pr(p_{\Delta_j} \wedge p_{\Delta_i})}{\Pr(p_{\Delta_i})}, \\ p_{\Delta_j} &\equiv \{r_j \in \Delta_j \subset \mathbb{R}\}, \quad p_{\Delta_i} \equiv \{r_i \in \Delta_i \subset \mathbb{R}\},\end{aligned} \tag{7}$$

which is well defined if $\Pr(p_{\Delta_i}) \neq 0$. To be consistent, we must verify that the expression just defined satisfies the probability conditions (i) (ii) and (iii).

It is obvious that $\Pr(p_{\Delta_j}|p_{\Delta_i}) \geq 0$, and therefore condition (i) is verified.

Let us consider that Δ_j and Δ'_j are two disjoint subsets of $\mathbb{R}$ ($\Delta_j \cap \Delta'_j = \phi$). Therefore the propositions $p_{\Delta_j} \equiv \{r_j \in \Delta_j\}$ and $p_{\Delta'_j} \equiv \{r_j \in \Delta'_j\}$ satisfy $p_{\Delta_j} \wedge p_{\Delta'_j} = \phi$. Let us consider

$$\begin{aligned}\Pr(p_{\Delta_j} \vee p_{\Delta'_j} | p_{\Delta_i}) &\equiv \frac{\Pr([p_{\Delta_j} \vee p_{\Delta'_j}] \wedge p_{\Delta_i})}{\Pr(p_{\Delta_i})} \\ &= \frac{\Pr([p_{\Delta_j} \wedge p_{\Delta_i}] \vee [p_{\Delta'_j} \wedge p_{\Delta_i}])}{\Pr(p_{\Delta_i})} \\ &= \frac{\Pr(p_{\Delta_j} \wedge p_{\Delta_i}) + \Pr(p_{\Delta'_j} \wedge p_{\Delta_i})}{\Pr(p_{\Delta_i})},\end{aligned}$$

where the last term follows from the fact that

$$[p_{\Delta_j} \wedge p_{\Delta_i}] \wedge [p_{\Delta'_j} \wedge p_{\Delta_i}] = (p_{\Delta_j} \wedge p_{\Delta'_j}) \wedge p_{\Delta_i} = \phi \wedge p_{\Delta_i} = \phi.$$

Therefore $\Pr(p_{\Delta_j} \vee p_{\Delta'_j} | p_{\Delta_i}) = \Pr(p_{\Delta_j} | p_{\Delta_i}) + \Pr(p_{\Delta'_j} | p_{\Delta_i})$, and we have verified condition (ii).

Condition (iii) is easily verified, as it is self evident from the following equation

$$\Pr(p_{\mathbb{R}} | p_{\Delta_i}) = \frac{\Pr(p_{\mathbb{R}} \wedge p_{\Delta_i})}{\Pr(p_{\Delta_i})} = \frac{\Pr(p_{\mathbb{R} \cap \Delta_i})}{\Pr(p_{\Delta_i})} = \frac{\Pr(p_{\Delta_i})}{\Pr(p_{\Delta_i})} = 1.$$

We emphasize that the consistency of the definition of the conditional probability given in (7) relies strongly on the fact that it is applied to propositions within a fixed basis of the Hilbert space. This is precisely the case in this paper, where we deal with propositions corresponding to the possible results of consecutive measurements.

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